Exercise 29

Let

$$f(x) = \begin{cases} \sqrt{-x} & \text{if } x < 0 \\ 3 - x & \text{if } 0 \le x < 3 \\ (x-3)^2 & \text{if } x > 3 \end{cases}$$

(a) Evaluate each limit, if it exists.

(i) $\lim_{x\to 0^+} f(x)$ (ii) $\lim_{x\to 0^-} f(x)$ (iii) $\lim_{x\to 0} f(x)$

(iv) $\lim_{x\to 3^-} f(x)$ (v) $\lim_{x\to 3^+} f(x)$ (vi) $\lim_{x\to 3} f(x)$

(b) Where is f discontinuous?

(c) Sketch the graph of f.

Solution

Evaluate each of the limits.

(i) $\lim_{x\to 0^+} f(x) = \lim_{x\to 0}(3 - x) = 3 - 0 = 3$

(ii) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0} \sqrt{-x} = \sqrt{0} = 0$

(iii) $\lim_{x\to 0} f(x) =$ Does not exist because left-hand and right-hand limits are unequal

(iv) $\lim_{x\to 3^-} f(x) = \lim_{x\to 3}(3 - x) = 3 - 3 = 0$

(v) $\lim_{x\to 3^+} f(x) = \lim_{x\to 3}(x-3)^2 = (3-3)^2 = 0$

(vi) $\lim_{x\to 3} f(x) = 0$

$\sqrt{-x}$, $3 - x$, and $(x-3)^2$ are all continuous on the intervals they're defined on, since the first is a root function and the last two are polynomial functions. As a result, any points of discontinuity can only occur at the endpoints, $x = 0$ and $x = 3$, of the intervals. The function is discontinuous at $x = 3$ because

$$\lim_{x\to 3} f(x) \neq f(3) = \text{undefined},$$

but the function is discontinuous at $x = 0$ because

$$\lim_{x\to 0} f(x) \neq f(0) = 3.$$

These results are confirmed in the graph of $f(x)$ versus x.